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Supercritical Ordered Trajectories with Quadratically Irrational Winding Number

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The dynamics of the circle map is studied in the supercritical regime where the map is not invertible and thus the trajectory elements are clustered on the circle. Existence of a simple ordering structure is established for trajectories with arbitrary irrational winding number. A previously developed formalism is then generalized to predict the trajectories when the winding number is quadratically irrational. Explicit results are given for a simple case.

KEY WORDS: Supercritical; ordered trajectory; universality; circle map.

1. INTRODUCTION

In the investigation of the behavior of dynamical systems one approach is to study maps from a low-dimensional manifold into itself. Such a map can be looked upon as discrete time evolution of a system or as a Poincaré map of some low-dimensional surface transverse to a periodic orbit in the phase space.

One such map that has been studied in the context of quasiperiodic route to chaos is the circle map

$$f(z) = z + \Omega - (k/2\pi)\sin(2\pi z) \tag{1.1}$$

defined on the whole real line—which is then assigned the topology of a circle by identifying z with $(z+1)$. This map describes motion on a two-torus $(c' \times c')$ with z_n as the nth point of intersection of the orbit with some cross section of the toroid. A useful parameter of the orbit generated by such a map, starting from the point z_0, is its winding number defined by

$$\omega(\Omega, k; z_0) = \lim_{n\to\infty} \frac{1}{n}\,[f^n(z_0) - z_0] \tag{1.2}$$

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The dynamics with $k=K$ and $k=-K$ are equivalent to a simple reparametrization of the circle. So we shall treat k as being positive always. In the domain $k<1$ the map is a diffeomorphism. At $k=1$ the inverse exists but has a cubic singularity at $z=0$ (or equivalent points). In appropriate contexts these cases have been studied by Shenker[1] and others.[2,3] My concern in this paper is the supercritical regime $k>1$ where the derivative of the map is less than zero for some ranges of z and thus the map is no longer invertible.[4,7]

The supercriticality of the map introduces complications into the possible dynamical behavior of the system. For example, the trajectory is no longer guaranteed to be "ordered." Ordering of a trajectory with some winding number W means that the relative locations of the actual trajectory points on the circle is the same as the one when the map is just a rotation by W. However, Kadanoff[5] (which will henceforth be referred to as SI) recently isolated a subclass of such ordered trajectories even in the supercritical domain, and for a particular winding number, the inverse of the golden mean $[=(\sqrt{5}-1)/2]$, he showed how to make detailed predictions about the locations of the trajectory points by using a kind of recursive analysis. Three kinds of trajectories, characterized by the behavior of the map at the end points of the interval containing it, were considered: ones with (i) both ends singular (slope of the map vanishing), (ii) only one end singular, and (iii) neither end singular.

This paper has four main objectives: (a) To show how the doubly singular trajectory may arise in connection with the circle map. This is done in the latter half of Section 2.1. (b) Section 2.2 shows that the class structure exists for trajectories with the winding number of not only the inverse golden mean but any arbitrary irrational value. (c) We also show explicitly in Section 3 how to predict the trajectories in the simple case where the winding number is of the form $\langle ppp... \rangle$, where $p \geqslant 2$. (d) Finally Section 4 briefly delineates the extension to the case of any quadratically irrational number. The conclusion includes some remarks for the golden mean case.

2. FORMULATION

2.1. General

The first half of this subsection is a brief summary of the relevant parts of SI just to set the context. However, it will be assumed that the reader who wants a detailed understanding is familiar with the contents of SI.

Let us consider the following family of maps:

$$R_{\Omega}(z)=R_0(z)+\Omega \tag{2.1}$$

where

$$R_0(z) = z - \left(\frac{k}{2\pi}\right) \sin(2\pi z) \tag{2.2}$$

For $k \leqslant 1$ all such maps are monotonic and ordering is trivially satisfied for any trajectory. However for $k > 1$ (see Fig. 1) there are sections of the map where the derivative is negative. There can be no elements of an ordered trajectory in the negative slope domain. So if there exists any ordered trajectory, the points in it must lie entirely in the closure of the positive slope domain. The existence of such trajectories, of any winding number, can easily be seen.

The argument starts by substituting the negative slope sections of the map by flat segments in a way that preserves the periodicity condition

$$f(z+1) = f(z) + 1 \tag{2.3}$$

Figure 2 shows how this can be achieved. Now consider the trajectory starting from X_0, the left edge of the positive slope domain that remains after the insertion of the flat segment. No other elements of this trajectory can fall in the closure of the flat segment if the winding number has to be irrational. Trajectory of any winding number can be generated by choosing Ω suitably and ordering is guaranteed since the new map is monotonic. This flattened

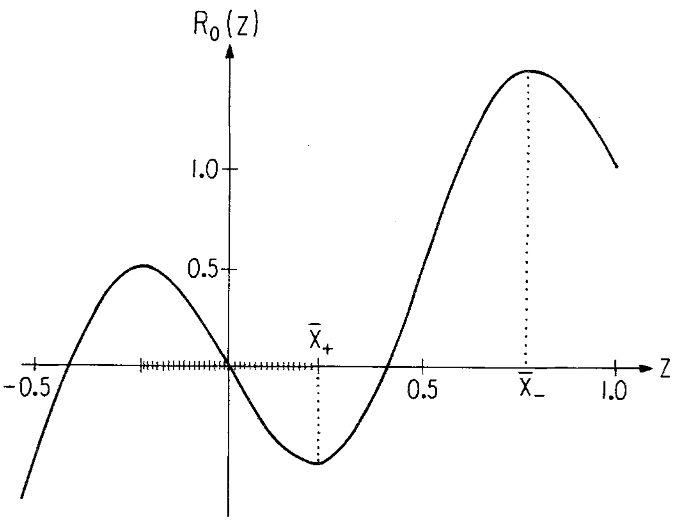

Fig. 1. Elements of an ordered trajectory cannot fall in the negative slope region, marked by crossed vertical bars (or shifted by an arbitrary integer).

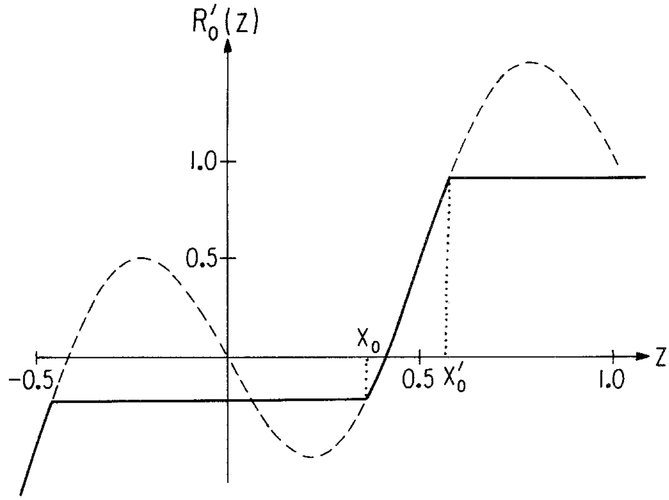

Fig. 2. Monotonizing the map in a way such that the allowed domain will have a positive nonvanishing derivative everywhere.

map is different from the original map only over the open set containing the flat regions—which, however, has no trajectory points in it anyway. So the trajectories belong to the original map as well.

For trajectories having irrational winding number the inverse is well defined and thus one can construct z_j for negative j values. In addition one can start with the point X_0', the right edge of the allowed positive slope domain, and construct yet another trajectory. To get the relationship between these two trajectories, let X_j and X_j' stand for the fractional parts of z_j and z_j', respectively. For any real z, $\{z\}$, the fractional part of z, is defined by $\{z\} \equiv$ (z − greatest integer less than z) and thus defines the location of z on the circle. The X_j and X_j' obey the following conditions:

(a) all elements fall into $[X_0, X_0']$;
(b) for $j > 0$, $X_j = X_j'$;
(c) for $j < 0$, $X_j' < X_j$;
(d) if $j < 0$ $[X_j', X_j]$ is a forbidden region;
(e) if $j \neq k$ the relative ordering of X_j and X_k' is the same as that of $\{wj\}$ and $\{wk\}$ (w is the winding number).

From Figure 2 one can see that the allowed positve slope domain can be categorized into two classes depending on how the flat segment is inserted into the map:

(i) *The general case*: where the derivative of the map is nonvanishing and positive everywhere (outside the flat interval), including both ends of the interval (as illustrated in Figure 2).

(iii) *The two special cases*: where the derivative vanishes at the left or the right end of the nonflat interval but nowhere else. Figures 3 and 4 illustrate these two situations.

However, there is one more possibility beyond the situations discussed above. For $k > 1$ the map

$$R_0(z) = z - \left(\frac{k}{2\pi}\right) \sin(2\pi z)$$

has two families of critical points (where the derivative vanishes) at (Fig. 1)

$$z = \bar{X}_\pm + \text{integers} \tag{2.4}$$

where

$$\cos(2\pi \bar{X}_\pm) = 1/k \tag{2.5}$$

Choose $\bar{X}_+$ to be the point of local minima in the range [0, 1]. Now choose k such that

$$\left(\frac{k}{2\pi}\right) \sin(2\pi \bar{X}_+) - \bar{X}_+ = (M+1)/2 \tag{2.6}$$

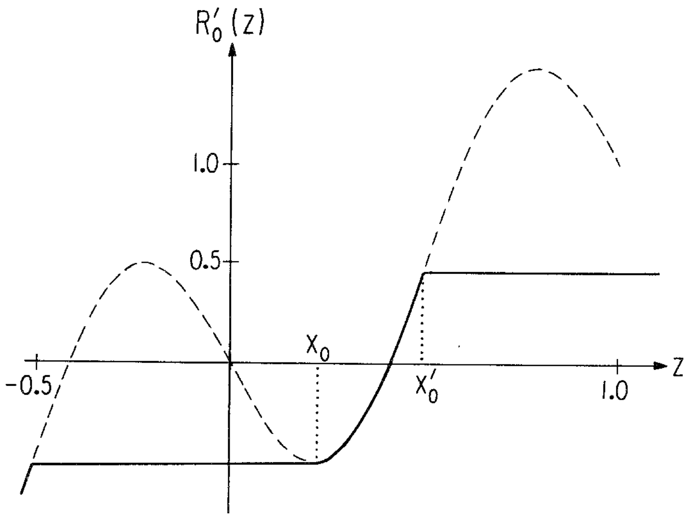

Fig. 3. The monotonized map is quadratic at the left end and linear at the right end of the allowed domain.

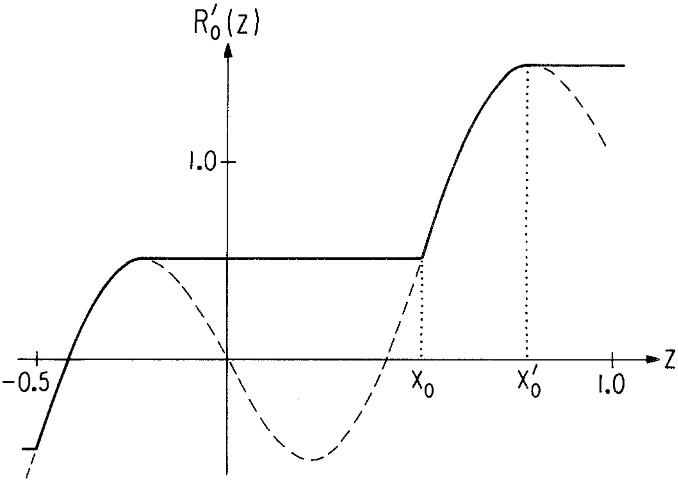

Fig. 4. Here the monotonized map is linear at the left end and quadratic at the right end of the allowed domain.

where M belongs to the set of nonnegative integers. Equations (2.5) and (2.6) have solutions for all such values of M. Now the height of the local maxima in $[0, 1]$ is

$$R_0(\bar{X}_-) = \bar{X}_- - \left(\frac{k}{2\pi}\right) \sin(2\pi\bar{X}_-) \tag{2.7}$$

and the height of the local minima in the range $[2 + M, 3 + M]$ is

$$R_0(\bar{X}_+ + 2 + M) = 2 + M + R_0(\bar{X}_+) \tag{2.8}$$

Now since $\bar{X}_- = 1 - \bar{X}_+$ notice that these two heights become identical when Eq. (2.6) is satisfied. Figure 5, which is a plot of the map $R_0(z)$ [and $R_0'(z)$] with a special value of k ($\simeq 4.604...$) satisfying both (2.5) and (2.6) for $M = 0$, illustrates this situation. In general the implication is that the height of the graph representing the map at the local maxima in $[0, 1]$ is the same as the height of the local minima in $[2 + M, 3 + M]$. As a result we can now insert a flat segment between these two points and extend this process so that the new map satisfies

$$R_0'(z + M + 2) = R_0'(z) + (M + 2) \tag{2.9}$$

The periodicity of this map R_0' is $(M + 2)$ rather than 1. By trivial extensions of the proofs given in SI it is possible to show that the winding number for this new family of maps

$$R_\Omega'(z) = R_0'(z) + \Omega$$

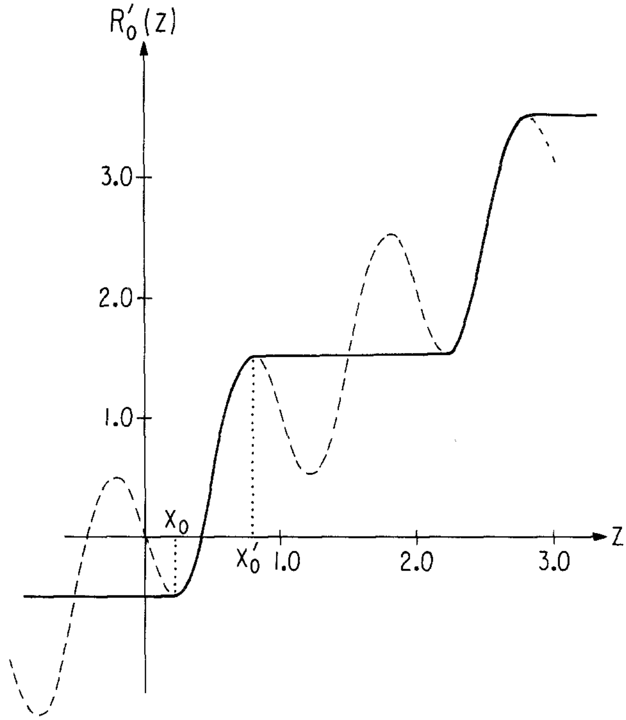

Fig. 5. Example of how doubly singular trajectory can be generated. The periodicity of the modified map is 2.

is independent of the initial point and a continuous, monotonically nondecreasing function of Ω. Also since now $w(\Omega + M + 2) = w(\Omega) + (M + 2)$, $w(\Omega)$ takes all possible values. Similarly the proof for ordering also goes through.

Just as in the two cases discussed at the beginning one can construct two trajectories starting from the left and the right ends of any connected allowed domain and all the earlier statements regarding the locations of the trajectory points continue to be valid. We have thus generated a new class of trajectories—since the derivative is now always positive but goes to zero at both ends—unlike the two cases discussed earlier.

2.2. Ordering Structure for Arbitrary Irrational Winding Number

An important step towards calculating the locations of the trajectory points for a nontrivial map is to identify the grouping pattern in it. Now because of ordering this reduces to the problem of recognizing the pattern for pure rotation with the same winding number, namely, to see how the points $\{wn\}$ (where n is integral) are grouped in the interval $[0, 1]$. We shall consider only positive irrational numbers less than one. A trajectory with a winding number that can be obtained by adding an integer to such a number is trivially related to the original one. Now any irrational number less than one can be written as

$$w = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{2.10}$$

where the a_i's $(i = 1, 2, 3,...)$ are positive integers. A more compact notation is

$$w = \langle a_1\, a_2\, a_3 \,...\rangle$$

To classify n's according to the fractional part of wn we introduce the following optimal sequence of rational approximations to w. The jth member $(j = 1, 2, 3,...)$ of the sequence is N_j/K_j, where $N_1 = 0$, $N_2 = 1$, $K_1 = 1$, $K_2 = a_1$, and

$$\left.\begin{aligned} N_{j+1} &= a_j N_j + N_{j-1} \\ K_{j+1} &= a_j K_j + K_{j-1} \end{aligned}\right\} \qquad \text{for} \quad j \geqslant 2 \tag{2.11}$$

Any arbitrary positive $(\geqslant 1)$ integer N can uniquely be represented as

$$N = \sum_{n=1}^{\infty} \beta_n(N) K_n \tag{2.12}$$

where $\beta_n(N)$ are subject to the following restrictions:

(i) $0 \leqslant \beta_1(N) \leqslant (a_1 - 1)$

(ii) $0 \leqslant \beta_n(N) \leqslant a_n \qquad$ for $n \geqslant 2$

(iii) $\beta_n = 0 \qquad$ if $\beta_{n+1} = a_{n+1}$

$c(N)$, the smallest value of n such that $\beta_n(N) \neq 0$ is called the class of N and $\beta_{c(N)}(N)$ is called the subclass of N. N is said to belong to the group $(c(N), \beta_{c(N)}(N))$. It turns out that the class and the subclass determines the grouping of the elements $\{wN\}$. To see it define $\Delta_n = wK_n - N_n$ so that $\Delta_n \gtrless 0$ for n odd or n even and $|\Delta_n| \to 0$ as $n \to \infty$. One also gets

$$\Delta_{n+1} = a_n \Delta_n + \Delta_{n-1} \tag{2.13}$$

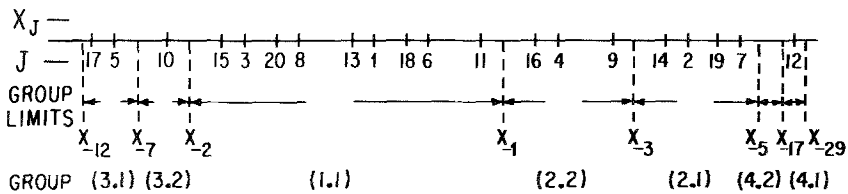

Fig. 6. Group structure for the trivial rotation map with the winding number of $w(2) = (\sqrt{2} - 1)$.

Now

$$\{wN\} = \left\{ \sum_{n=1}^{\infty} \beta_n(N) \Delta_n \right\} \tag{2.14}$$

By employing the restrictions of β_n as mentioned earlier one gets, after some algebra, the following results (see Fig. 6). All elements $\{wN\}$ with N belonging to a specific group densely cover a continuous interval. Even class index increases from the center to the right whereas within a given even class, the subclass index increases from right to left. For odd classes everything is reversed. Every group is bounded at the ends by some negative N element of the trajectory. Table I summarizes the results, showing the group limits as well as the negative elements that bound them. Figure 6 illustrates the results for the case where $w = \langle 22222 \ldots \rangle$.

The basic reason why this analysis of grouping pattern is useful is that for the supercritical case all the points within a given group will be tighly bunched together leaving wide gaps between adjacent groups. As a result all the points within a given group will be assigned a single coordinate and it will be this coordinate that we shall predict. We shall do it explicitly in the next section for the simple case $w = \langle ppp \ldots \rangle$ for $p \geqslant 2$. This is simple in the sense that it is periodic right from the beginning and has periodicity one. Hints as to how the analysis can be extended to the cases where these restrictions are removed are given in Section 4. Notice that in the next section P_i's have replaced K_i's.

3. RECURSIVE ANALYSIS FOR THE WINDING NUMBER $\langle pppp \ldots \rangle$ $(p \geqslant 2)$

3.1. Formulation

We start with the following two basic observations:

(a) All the elements in a group are tightly bunched together.

Table I. Ordering Structure and Group Limits

		Negative bounding element			
Class	Left limit[a]	Left	Right	Right limit[b]	Width
1	$(p-1)\Delta_1-\Delta_2$	$K_2-(p-1)K_1$	K_2-pK_1	$p\Delta_1-\Delta_2$	Δ_1
		$1 \leqslant p \leqslant (a_1-1)$			
$(2m+1)$	$(p-1)\Delta_{2m+1}-\Delta_{2m+2}$	$K_{2m+2}-(p-1)K_{2m+1}$	$K_{2m+2}-pK_{2m+1}$	$p\Delta_{2m+1}-\Delta_{2m+2}$	Δ_{2m+1}
$m=1, 2, 3,\ldots$		$1 \leqslant p \leqslant a_{2m+1}$			
$2m$	$1+p\Delta_{2m}-\Delta_{2m+1}$	$K_{2m+1}-pK_{2m}$	$K_{2m+1}+(1-p)K_{2m}$	$1+(p-1)\Delta_{2m}-\Delta_{2m+1}$	$-\Delta_{2m}$
$m=1, 2, 3,\ldots$		$1 \leqslant p \leqslant a_{2m}$			

[a] Left limit $= X_{-\text{negative bounding element(left)}}$
[b] Right limit $= X_{-\text{negative bounding element(right)}}$

(b) If $\alpha \geqslant 3$ then any two points in the group $G(\alpha, \beta)$ will have the same group history over the next $(P_{\alpha-1} - 1)$ steps and $(P_\alpha - 1)$ steps for $\beta = p$ and $1 \leqslant \beta \leqslant (p-1)$, respectively. And this group history is the same as that of the corresponding number of iterates of z_0 (or X_0 or X_0').

In the course of $(P_c - 1)$ iterations of z_0 it is P_{c-l} times in each of the subclasses of class l for $1 \leqslant l \leqslant (c-1)$. If $b_{(\alpha,\beta)}$ is the derivative of the map in group $G(\alpha, \beta)$ then let us define

$$D_c = \prod_{\alpha=1}^{c-1} \prod_{\beta} b_{(\alpha,\beta)}^{P_{c-\alpha}} \qquad \text{for} \quad c \geqslant 2 \tag{3.1}$$

Notice that D_c is just the product of the derivatives along the trajectory and hence is essentially a sensitivity factor that determines how far two initially close points in group $G(\alpha, \beta)$ will get separated after the iteration process. Also define $D_1 = 1$ for later notational convenience.

The recursion relation that D_c satisfies is

$$D_{c+1} = D_c^p D_{c-1} \left[\prod_s b_{(c,s)} \right] \qquad \text{for} \quad c \geqslant 3 \tag{3.2}$$

With the definition $H_c = \prod_s b_{(c,s)}$ this becomes

$$D_{c+1} = D_c^p D_{c-1} H_c \tag{3.3}$$

Now set the notations

$$X_\alpha^{(2m+1)} = \{z_{\alpha P_{2m+1}}\} - X_0 = X_{\alpha P_{2m+1}} - X_0 \tag{3.4}$$

and

$$Y_\alpha^{(2m)} = X_0' - \{z_{\alpha P_{2m}}\} = X_0' - X_{\alpha P_{2m}} \tag{3.5}$$

which are just the distances of the groups $(2m+1, \alpha)$ and $(2m, \alpha)$ from X_0 and X_0', the left and right ends of the allowed interval.

Let

$$\varepsilon_\alpha^{(2m+1)} = R(X_{\alpha P_{2m+1}}) - R(X_0) \tag{3.6}$$

Also define

$$Q_\alpha^{(2m+1)} = R^{P_{2m+1}}(X_{\alpha P_{2m+1}}) - R^{P_{2m+1}}(X_0) \qquad \text{for} \quad 1 \leqslant \alpha \leqslant (p-1) \tag{3.7}$$

and

$$Q_p^{(2m+1)} = R^{P_{2m}}(X_{pP_{2m+1}}) - R^{P_{2m}}(X_0) \tag{3.8}$$

From our observations about the group history of iterations and the definitions made earlier one sees that for $1 \leqslant \alpha \leqslant (p-1)$

$$Q_\alpha^{(2m+1)} = \varepsilon_\alpha^{(2m+1)} D_{2m+1} \tag{3.9}$$

where use has been made of a linear analysis. Also

$$Q_\alpha^{(2m+1)} = X_{(\alpha+1)}^{(2m+1)} - X_1^{(2m+1)} \tag{3.10}$$

Now if we make use of the assumption that the groups converge very fast to the ends, $X_{(\alpha+1)}^{(2m+1)} \gg X_1^{(2m+1)}$ and thus

$$\varepsilon_\alpha^{(2m+1)} D_{2m+1} = Q_\alpha^{(2m+1)} = X_{(\alpha+1)}^{(2m+1)} \qquad \text{for} \quad 1 \leqslant \alpha \leqslant (p-1) \tag{3.11}$$

Similarly

$$\varepsilon_p^{(2m+1)} D_{2m} = Q_p^{(2m+1)} = Y_1^{(2m)} \tag{3.12}$$

For even classes we follow an identical procedure and thus define

$$\varepsilon_\alpha^{(2m)} = R(X_0') - R(X_{\alpha P_{2m}}) \tag{3.13}$$

$$Q_\alpha^{(2m)} = R^{P_{2m}}(X_0') - R^{P_{2m}}(X_{\alpha P_{2m}}) \qquad \text{for} \quad 1 \leqslant \alpha \leqslant (p-1) \tag{3.14}$$

and

$$Q_p^{(2m)} = R^{P_{2m-1}}(X_0') - R^{P_{2m-1}}(X_{pP_{2m}}) \tag{3.15}$$

And now as above one gets

$$\varepsilon_\alpha^{(2m)} D_{2m} = Q_\alpha^{(2m)} = Y_{(\alpha+1)}^{(2m)} \qquad \text{for} \quad 1 \leqslant \alpha \leqslant (p-1) \tag{3.16}$$

and

$$\varepsilon_p^{(2m)} D_{2m-1} = Q_p^{(2m)} = X_1^{(2m-1)} \tag{3.17}$$

In order to calculate the widths of the groups note that

$$\Delta X_{2m,\alpha} = X'_{-[P_{2m+1}-(\alpha-1)P_{2m}]} - X_{-[P_{2m+1}-\alpha P_{2m}]} \tag{3.18}$$

For $\alpha = p$ this becomes

$$\Delta X_{2m,p} = X'_{-[P_{2m}+P_{2m-1}]} - X_{-P_{2m-1}} \tag{3.19}$$

Now iterate the ends of the group P_{2m-1} times. Since the groups are very tightly bunched once again linear analysis is applicable and we get

$$\Delta X_{2m,p} b_{(2m,p)} D_{2m-1} = X'_{-P_{2m}} - x_0 = X_p^{(2m+1)} \tag{3.20}$$

where we employed the facts that $X'_{-P_{2m}}$ constitutes one boundary of the group $(2m+1, p)$ and that the width of a group is very small compared to its distance from the appropriate end.

For $1 \leqslant \alpha \leqslant (p-1)$, to get the group width we carry out the iteration P_{2m} times to get

$$\Delta X_{2m,\alpha} b_{(2m,\alpha)} D_{2m} = \Delta X_{2m,(\alpha+1)} \tag{3.21}$$

Thus one can calculate the width of a group from the width of the next higher group in the same class.

Analogous analysis applies to the groups with odd class. One gets

$$\Delta X_{(2m+1),p} b_{(2m+1),p} D_{2m} = Y_p^{(2m+2)} \tag{3.22}$$

and

$$\Delta X_{(2m+1),\alpha} b_{(2m+1),\alpha} D_{2m+1} = \Delta X_{(2m+1),(\alpha+1)} \tag{3.23}$$

Equations (3.2), (3.11), (3.12), (3.16), and (3.17), along with the definitions of the variables involved in these equations, incorporate all the information that is needed for a recursive analysis of the trajectory. Once that is done equations (3.20), (3.21), (3.22), and (3.23) can be utilized to calculate the group widths.

Before we analyze the three universality classes separately we note that in all the three cases the map has a polynomial form at the ends. Thus, $\varepsilon_s^{(2m)} = \theta[Y_s^{2m}]^{k_1}$, and $\varepsilon_s^{(2m+1)} = \phi[X_s^{(2m+1)}]^{k_2}$. For the quadratic end $k_i = 2$ ($i = 1$ and/or 2) and for the linear end $k_i = 1$ ($i = 1$ and/or 2). This implies that

$$b_{(2m,\alpha)} Y_\alpha^{(2m)} = k_1 \varepsilon_\alpha^{(2m)} \quad \text{and} \quad b_{((2m+1),\alpha)} X_\alpha^{(2m+1)} = k_2 \varepsilon_\alpha^{(2m+1)}$$

So from Eqs. (3.17) and (3.20)

$$\frac{\Delta X_{2m,p}}{Y_p^{(2m)}} = \frac{X_p^{(2m+1)}}{k_1 X_1^{(2m-1)}}$$

which, ignoring k_1, is just the ratio of the distances of two neighboring odd class groups from the left end. Similarly

$$\frac{\Delta X_{2m,\alpha}}{Y_\alpha^{(2m)}} = \frac{\Delta X_{2m,(\alpha+1)}}{k_1 Y_{(\alpha+1)}^{(2m)}}$$

and

$$\frac{\Delta X_{(2m+1),p}}{X_p^{(2m+1)}} = \frac{Y_p^{(2m+2)}}{k_2 Y_1^{(2m)}}, \qquad \frac{\Delta X_{(2m+1),\alpha}}{X_\alpha^{(2m+1)}} = \frac{\Delta X_{(2m+1),(\alpha+1)}}{k_2 X_{(\alpha+1)}^{(2m+1)}}$$

The important implication of this last group of results is the following: if the analysis of the trajectory points shows that the groups indeed converge to the respective ends very fast that also ensures the validity of the other basic assumption that the width of a group is very small compared to its distance from the relevant end.

This also shows that, asymptotically, at the linear end this ratio of the group width to the group location is the same for all groups within a given class. However, at the quadratic end this ratio falls geometrically (by a factor of 2 with each group) with group index decreasing.

3.2. Results for the Three Classes

(a) Generic Case—No Extrema. Since the map is linear at both ends

$$
\begin{aligned}
b_{c,s} &\to b \qquad \text{for } c \text{ odd} \\
&\to b' \qquad \text{for } c \text{ even}
\end{aligned}
$$

Then the recursion relation (3.2) becomes

$$\ln D_{2m+1} = p \ln D_{2m} + \ln D_{2m-1} + p \ln b' \tag{3.24}$$

and

$$\ln D_{2m+2} = p \ln D_{2m+1} + \ln D_{2m} + p \ln b \tag{3.25}$$

The solution to Eqs. (3.24) and (3.25) can immediately be seen to be[6]

$$\ln D_{2m} = -\ln b' + A_+ \omega^{-2m} + A_-(-\omega)^{2m} \tag{3.26}$$

and

$$\ln D_{2m+1} = -\ln b + A_+ \omega^{-(2m+1)} + A_-(-\omega)^{2m+1} \tag{3.27}$$

Here A_+ and A_- are nonuniversal with $A_+ > 0$ which ensures that the groups converge to the ends.

From this finding out the group locations is just a matter of some tedious algebra. The results are

$$
\begin{aligned}
\ln X_\alpha^{(2m+1)} = \theta_0 &- A_+[(1+\omega) + (p-\alpha)]\omega^{-(2m+1)} \\
&- A_-\left[\left(\frac{1}{\omega} - 1\right) - (p-\alpha)\right]\omega^{(2m+1)}
\end{aligned} \tag{3.28}
$$

and

$$\ln Y_{\alpha}^{(2m)} = \ln\left(\frac{be^{\theta_0}}{b'}\right) - A_{+}[(1+\omega)+(p-\alpha)]\,\omega^{-2m}$$
$$-A_{-}\left[\left(1-\frac{1}{\omega}\right)+(p-\alpha)\right]\omega^{2m} \tag{3.29}$$

where θ_0 is nonuniversal again. Thus this universality class is characterized by three free parameters—θ_0, A_+, and A_-.

(b) Both Ends Quadratic. For large m

$$\varepsilon_{\beta}^{(2m+1)} \to \left[\frac{X_{\beta}^{(2m+1)}}{\lambda_1}\right]^2$$

$$b_{(2m+1,\beta} \to \left[\frac{2X_{\beta}^{2m+1)}}{\lambda_1^2}\right]$$

and

$$\varepsilon_{\beta}^{(2m)} \to \left[\frac{Y_{\beta}^{(2m)}}{\lambda_2}\right]^2$$

$$b_{2m,\beta} \to \left[\frac{2Y_{\beta}^{(2m)}}{\lambda_2^2}\right]$$

Incorporating these into Eqs. (3.11), (3.12), (3.16), and (3.17) one gets

$$b_{(2m+1),(\alpha+1)} = \tfrac{1}{2}b_{(2m+1),\alpha}^2 D_{2m+1} \tag{3.30}$$

$$b_{2m,1} = (\lambda_1^2/2\lambda_2^2)\, b_{(2m+1),p}^2 D_{2m} \tag{3.31}$$

$$b_{2m,(\alpha+1)} = \tfrac{1}{2}b_{2m,\alpha}^2 D_{2m} \tag{3.32}$$

and

$$b_{(2m-1),1} = (\lambda_2^2/2\lambda_1^2)\, b_{2m,p}^2 D_{2m-1} \tag{3.33}$$

Recursively applying (3.30) and (3.32) to eliminate all $b_{c,\alpha}$ for $2 \leqslant \alpha \leqslant (p-1)$ we get

$$b_{(2m+1),p} = (D_{2m+1}/2)^{2^{p-1}-1}\, b_{(2m+1),1}^{2^{p-1}} \tag{3.34}$$

and

$$b_{2m,p} = (D_{2m}/2)^{2^{p-1}-1}\, b_{2m,1}^{2^{p-1}} \tag{3.35}$$

Notice that the basic goal of the algebra involved is to extract a recursion relation among variables of a given type only, let us say $b_{c,\alpha}$, where α is fixed. To achieve this elimination of variables we calculate, using Eqs. (3.30)–(3.33), H_{2m} and $H_{2m}+1$ to be given by

$$H_{2m}=\frac{b^2_{2m,p}}{(D_{2m}/2)^{p-1}b_{2m,1}} \tag{3.36}$$

and

$$H_{2m+1}=\frac{b^2_{(2m+1),p}}{(D_{2m+1}/2)^{p-1}b_{(2m+1),1}} \tag{3.37}$$

Now just substitute the expressions for H_{2m} and H_{2m+1} from (3.36) and (3.37) in the recursion relation (3.3) to get

$$D_{2m+2}b_{(2m+1),1}=4^pD_{2m}b_{(2m-1),1} \tag{3.38}$$

and

$$D_{2m+3}b_{(2m+2),1}=4^pD_{2m+1}b_{2m,1} \tag{3.39}$$

The next step is to cyclically use (3.30)–(3.33) to get a relation between $b_{c,1}$ and $b_{(c+2),1}$ in terms of λ_1, λ_2 and the D's only. Thus we get

$$b_{(2m-1),1}=f(\lambda_1,\lambda_2,p)\,D_{2m-1}D_{2m}^{2^{p+1}-2}D_{2m+1}^{2^{2p}-2^{p+1}}b_{(2m+1),1}^{2^{2p}} \tag{3.40}$$

and

$$b_{2m,1}=g(\lambda_1,\lambda_2,p)\,D_{2m}D_{2m+1}^{2^{p+1}-2}D_{2m+2}^{2^{2p}-2^{p+1}}b_{(2m+2),1}^{2^{2p}} \tag{3.41}$$

Here f and g are functions that depend only on the variables within the parentheses. So the final step towards eliminating the model-dependent parameters λ_1, λ_2 and the D's is simply to use (3.40) and (3.41) for two consecutive values of m and the fact that the ratio of D_c to D_{c+2} can be written in terms of p and the ratio of $b_{(c-1),1}$ to $b_{(c+1),1}$ only from Eqs. (3.38) and (3.39). Thus we finally get the desired recursion relation:

$$\begin{aligned}
p(2^{2p}-1)\ln 4 &\\
&= -\ln b_{c,1}-(2^{p+1}-1)\ln b_{(c+1),1}-(2^{2p}-2^{p+1}-1)\ln b_{(c+2),1}\\
&\quad +(2^{2p}+2^{p+1}-1)\ln b_{(c+3),1}+(2^{2p}-2^{p+1})\ln b_{(c+4),1}\\
&\quad -2^{2p}\ln b_{(c+5),1}
\end{aligned} \tag{3.42}$$

To get the solution for the homogeneous version (just put the left-hand side equal to zero) of this linear inhomogeneous difference equation put $\ln b_{c,1} = \phi\lambda^c$. Substituting into (3.42) the eigenvalue equation is found to be

$$(\lambda - 1)^2(\lambda + 1)(\lambda + 1/2^p)^2 = 0 \tag{3.43}$$

Thus the homogeneous solution is

$$\ln b_{c,1} = (\phi_1 + \phi_2 c) + \phi_3(-1)^c + (\phi_4 + \phi_5 c)(-1/2^p)^c \tag{3.44}$$

For the inhomogeneous solution try

$$\ln b_{c,1} = c^2\phi \tag{3.45}$$

One finds that (3.45) solves (3.42) with

$$\phi = -\frac{p(2^{2p} - 1)\ln 2}{2(1 + 2^p)^2} \tag{3.46}$$

The full solution to (3.42) is just the sum of the homogeneous and the inhomogeneous parts:

$$\ln b_{c,1} = (\phi_1 + \phi_2 c) + \phi_3(-1)^c + (\phi_4 + \phi_5 c)\left(-\frac{1}{2^p}\right)^c - \frac{p(2^{2p} - 1)\ln 2}{2(1 + 2^p)^2}c^2 \tag{3.47}$$

Since for large c the quadratic term (which has no free parameters in it) is the dominant one, all the free parameters are unrestricted. Once $b_{c,1}$ has been found out, deriving the explicit expression for $b_{c,\alpha}$ for $2 \leqslant \alpha \leqslant p$ needs only the knowledge of D_c which satisfies the recursion relation

$$D_c b_{(c-1),1} = 4^p D_{c-2} b_{(c-3),1} \tag{3.48}$$

the solution to which is trivially

$$\ln D_c = -\ln b_{(c-1),1} + cp \ln 2 + \xi \tag{3.49}$$

where ξ is another arbitrary constant. If in (3.47) we ignore the third term, which dies out with c increasing, direct substitution shows that

$$\ln(b_{c,\alpha}/b_{c,1}) = (2^{\alpha-1} - 1)\left[\frac{cp\ln 4}{(1 + 2^p)} + \xi + \phi_2 + 2\phi_3(-1)^c - \ln 2 + \frac{p(2^{2p} - 1)\ln 2}{2(1 + 2^p)^2}\right] \tag{3.50}$$

Since the dominant term is linear in c, positive for all $2 \leqslant \alpha \leqslant p$, and increases with α we are guaranteed that within a given class, the subclasses indeed converge to the appropriate ends with the subclass index going from p to 1.

(c) One End Quadratic. We shall treat only the situation where the left end of the map within the allowed interval is quadratic.

Because of the stated behavior of the map, for large m,

$$\varepsilon_\beta^{(2m+1)} \to [X_\beta^{(2m+1)}/\lambda]^2 \tag{3.51}$$

$$\varepsilon_\beta^{(2m)} \to bY_\beta^{(2m)} \tag{3.52}$$

$$b_{(2m+1),\beta} \to [2X_\beta^{(2m+1)}/\lambda^2] \tag{3.53}$$

$$b_{2m,\beta} \to b \tag{3.54}$$

Analogous to Eqs. (3.30)–(3.33) we have now

$$b_{(2m+1),(\alpha+1)} = (D_{2m+1}/2)b^2_{(2m+1),\alpha} \tag{3.55}$$

$$Y_1^{2m)} = (\lambda^2/4)D_{2m}b^2_{(2m+1),p} \tag{3.56}$$

$$Y_{(\alpha+1)}^{(2m)} = (bD_{2m})Y_\alpha^{(2m)} \tag{3.57}$$

and

$$(\lambda^2/2)b_{(2m-1),1} = bY_p^{(2m)}D_{2m-1} \tag{3.58}$$

Once again, as in the last case, the algebra proceeds in quite a similar way with the goal of eliminating variables so as to obtain a recursion relation among variables of a given type only. Hence we do not produce the details here. The recursion relation that we get as a result is

$$\begin{aligned}(2^p - 1)p^2 \ln 2 = \ln b_{(2m-3),1} - [(2^p - 1)p + 2 + 2^p] \ln b_{(2m-1),1} \\ + [p(2^p - 1) + 1 + 2^{p+1}] \ln b_{(2m+1),1} - 2^p \ln b_{(2m+3),1}\end{aligned} \tag{3.59}$$

the solution to which, in the way described earlier, can be seen to be

$$\ln b_{(2m+1),1} = \alpha + \frac{(2m+1)p \ln 2}{2} + a_1 X_1^m + a_2 x_2^m \tag{3.60}$$

where X_1 and X_2 are the greater than one and the less than one roots of the quadratic equation

$$2^p x^2 - [p(2^p - 1) + 1 + 2^p]X + 1 = 0 \tag{3.61}$$

Here $a_1 < 0$ but otherwise free and α and a_2 are unrestricted free parameters. We shall now show (instead of giving an explicit analytic form) that groups within a given class have the right convergence behavior. With the help of some algebra that we have not shown here we can write $\ln D_{2m+1}$ as

$$\ln D_{2m+1} = \ln 2 + \frac{1}{(2^p - 1)} \ln b_{(2m-1),1} - \frac{2^p}{2^p - 1} \ln b_{(2m+1),1} \quad (3.62)$$

A recursive application of (3.55) results in

$$\ln b_{(2m+1),\alpha} = (2^{\alpha-1} - 1) \ln D_{2m+1} - (2^{\alpha-1} - 1) \ln 2 + 2^{\alpha-1} \ln b_{(2m+1),1} \quad (3.63)$$

Now just substitute the expression for $\ln D_{2m+1}$ from (3.62) into (3.63) to get the desired result, which is

$$\ln[b_{(2m+1),\alpha}/b_{(2m+1),1}] = \left(\frac{2^{\alpha-1} - 1}{2^p - 1}\right) \ln[b_{(2m-1),1}/b_{(2m+1),1}] \quad (3.64)$$

Since $b_{(2m-1),1} > b_{(2m+1),1}$ [for a given subclass, higher classes are closer to the end], for $\alpha \geqslant 2$ the right-hand side is positive and an increasing function of α. This implies that $b_{(2m+1),\alpha}$ is greater than $b_{(2m+1),1}$ for $\alpha \geqslant 2$ and is a monotonically increasing function of α—which is what we want.

For the even class groups we have the following result:

$$Y_1^{(2m)} \sim 4^{mp} \exp\left[\sum_i a_i X_i^m (X_i 2^p - 1)/(X_i(2^p - 1))\right] \quad (3.65)$$

The expressions for the even class groups with subclass index other than one are involved. But it is fairly trivial to show that they have all the expected ordering and convergence properties.

4. GENERALIZATION TO THE CASE OF ARBITRARY QUADRATIC IRRATIONAL

The essential ingredients of our analysis in Section 3 were Eqs. (3.11), (3.12), (3.16), and (3.17) since they determine the group locations. And then Eqs. (3.20), (3.21), (3.22), and (3.23) give the group widths. A careful examination of the formalism developed there shows that this whole procedure goes through only if one generalizes the definition of D_c to be the sensitivity factor:

$$D_c = \prod_{i=1}^{K_c - 1} b_{(\alpha,\beta)(i)} \quad (4.1)$$

where $(\alpha, \beta)(i)$ stands for the group containing the ith iterate of z_0. The recursion relation changes to

$$D_{c+1} = D_c^{a_c} D_{c-1} \left[\prod_s b_{(c,s)} \right] \tag{4.2}$$

Definition (3.1) and Eq. (3.2) are just special cases of (4.1) and (4.2), respectively. So the set of observations that groups converge very fast to the respective end and that the width of a group is much smaller than its distance from the appropriate end allow us to write down all the equations necessary for the most general case. However, the obvious reason why we were able, in Section 3, to solve them explicitly is that the continued fraction representation $\langle pppp... \rangle$ has a finite (one in this case) periodicity and thus we have a finite set of difference equations. For a general quadratic irrational we shall have a similar situation. Thus we have in principle a way to approach the problem, although the task of solving the finite difference equations remains.

5. GENERAL REMARKS

The analysis as presented here has been self-consistent in the sense that the input of observations regarding the locations and the widths of the groups, which were made on the basis of numerical data and plausible guesses, is consistent with the predictions of the analysis. It would be desirable to be able to prove this input analytically. It may be pointed out here that the analysis in SI for the winding number $\langle 111... \rangle$ contains a fundamental mistake. The minimum number of steps for which the class history of two points in the class c will be identical is $(F_{c-1} - 1)$ and not $(F_c - 1)$ as noted there. This renders much of the quantitative results as obtained there incorrect.

We have numerical data that are extensive for the $\langle 1111... \rangle$ case and convincing but not good enough for the $\langle 222... \rangle$ case. The agreement between the data and our analytic predictions is excellent and barely convincing, respectively. The reason why it is difficult to obtain precise enough numerical data for the $\langle ppp... \rangle$ case with $p \geqslant 2$ is that the convergence of the groups to the ends is extremely fast. Even for $p = 2$ only, the quadrupole precision mode generates data that are barely convincing because the predictions hold only for large classes and error propagation makes the data meaningless at that level.

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REFERENCES

1. S. Shenker, *Physica* **5D**:405 (1982).
2. M. Feigenbaum, L. Kadanoff, and S. Shenker, *Physica* **5D**:370 (1982).
3. D. Rand, S. Ostlund, J. Sethna, and E. Siggia, *Phys. Rev. Lett.* **49**:132 (1982).
4. L. Glass and R. Perez, *Phys. Rev. Lett.* **48**:1772 (1982).
5. L. P. Kadanoff, *J. Stat. Phys.* **31**:1 (1983).
6. C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
7. T. Geisel and J. Nierwetberg, *Phys. Rev. Lett.* **48**:7 (1982).